

Exercises in Probability

A Guided Tour from Measure Theory to Random Processes,
via Conditioning

Second Edition

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CAMBRIDGE
UNIVERSITY PRESS

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