Randolph Nelson

Probability, Stochastic Processes, and Queuekig Theory

The Mathematics of Computer Performance Modeling

With 68 Figures

Springer

Contents

Preface **vii**

Acknowledgments **ix**

List of Figures **xxiii**

List of Tables **xxvii**

1 Introduction **1**

1.1 Outline of Book 3

1.1.1 Outline of Part I 3

1.1.2 Outline of Part II 5

1.1.3 Applied Probability and Performance Modeling 6

1.2 Queueing Systems** 8

1.2.1 More General Queueing Systems 9

1.2.2 Queueing Networks 10

I Probability 15

2 Randomness and Probability **17**

2.1 Probability: Frequency-Based 20

2.2 Probability: Axiornatic-Based . . . 22
2.2.1 Probability and Modeling . . . 23
2.2.2 Set Operations . . . 26
2.2.3 Probability Spaces . . . 28
2.3 Probability: Immediate Consequences of the Axioms . . . 32
2.3.1 Conditional Probability . . . 34
2.3.2 Independence . . . 36
2.3.3 Causation . . . 38
2.3.4 Bayes' Rule . . . 38
2.4 Summary of Chapter 2 . . . 39
2.4.1 Postscript: Calculating -*n* to a Billion Places . . . 41
2.5 Problems for Chapter 2 . . . 42

3 Combinatorics 45
3.1 Different Counting Paradigms . . . 47
3.1.1 Counting Methodology . . . 47
3.2 The Four Counting Paradigms . . . 49
3.2.1 Permutations without Replacement . . . 49
3.2.2 Permutations with Replacement . . . 50
3.2.3 Coincidences . . . 52
3.2.4 Combinations without Replacement . . . 54
3.2.5 Problems Equivalent to Combinations without Replacement . . . 56
3.2.6 Basic Properties of Binomial Coefficients . . . 58
3.2.7 Binomial Identities . . . 59
3.2.8 Multinomial Coefficients . . . 61
3.2.9 Partial Sum of Binomial Coefficients . . . 62
3.2.10 Combinations with Replacement . . . 62
3.2.11 Simple Derivation for Combinations with Replacement . . . 64
3.2..12 Problems Equivalent to Combinations with Replacement . . . 65
3.2.13 Basic Properties of Binomial-R Coefficients . . . 66
3.3 Generating Functions . . . 67

3.3.1 Generating Functions for Combinations . . . 68
3.3.2 General Solution for Combinations . . . 69
3.3.3 Algebraic Properties of Generating Functions . . . 74
3.3.4 Addition and Convolution . . . 74
3.3.5 First Passage Times . . . 76
3.4 Solving Recurrence Equations with Generating Functions** . . . 80
3.4.1 An Example Using Generating Functions . . . 82
3.4.2 General Generating Function Solution Method . . . 84
3.4.3 Solutions without Explicitly Determining the Coefficients . . . 88
3.5 Summary of Chapter 3 . . . 90
3.6 Problems for Chapter 3 . . . 92

4 Random Variables and Distributions 101

4.1 Random Variables . . . 102
4.1.1 Forms of Equality . . . 106
4.1.2 Independence . . . 106
4.1.3 Examples of Random Variables . . . 107
4.1.4 Joint Random Variables . . . 109
4.2 Distribution Functions . . . I11
4.2.1 Impulse Functions . . . 114
4.2.2 Probability Generating Functions . . . 116
4.2.3 Existence of Probability Spaces for a Given Distribution . . . 117
4.3 Unified Derivation of Common Distributions . . . 118
4.3.1 Indicator Random Variables . . . 119
4.3.2 Bernoulli Random Variables . . . 120
4.3.3 Discrete Uniform Distribution . . . 121
4.3.4 Properties of Indicator Random Variables . . . 122
4.4 Constructing Discrete Distributions from Indicator Random Variables . . . 123
4.4.1 Operations Used to Form Random Variables . . . 124
4.4.2 Discrete Uniform Random Variables . . . 124
4.4.3 Binomial Random Variables . . . 125

t

4.4.4 Sum of Binomial Random Variables with the Same Parameter 127
4.4.5 Intuitive Argument for Sum of Binomial Random Variables 127
4.4.6/ Sum of Binomial Random Variables with Different Parameters 128
4.4.7 Geometric Random Variables 129
4.4.8 Geometric Sum of Geometric Random Variables 130
.4.4.9 Starting Geometric Distributions from 0 131
4.4.10 Memoryless Property of the Geometric Distribution 131
4.4.11 Discrete 4- Memoryless ⇒ Geometric 132
4.4.12 Negative Binomial Distribution 133
4.4.13 Sum of Negative Binomial Random Variables 135
1 4.4.14 Starting Negative Binomial Distributions from 0 135
4.4.15 Relationship Between Geometric and Binomial Distributions 136
f 4.4.16 Renewal Counting Process of Geometric Random Variables 137
4.4.17 Intuitive Argument for the Renewal Counting Process 137
4.4.18 Relationship Between Uniform and Binomial Distributions 139
4.5 Limiting Distributions 140
4.5.1 Continuous Uniform Distribution 140
4.5.2 Probability of Zero ^- Impossibility 141
4.5.3 The Poisson Distribution 144
4.5.4 * Poisson Approximation of a Binomial Distribution 145
4.5.5 Sum of Poisson Random Variables 146
4.5.6 Generalized Limit that Leads to the Poisson Distribution 147
4.5.7 Difference Between Binomial and Poisson Random Variables 148
4.5.8 The Exponential Distribution 149
4.5.9 Continuous + Memoryless ⇒ Exponential 151
4.5.10 Geometric Sum of Exponential Random Variables 152
4.5.11 Renewal Counting Process of Exponential Random Variables 158
4.6 Summary of Chapter 4 159
4.7 Problems for Chapter 4 161

5 Expectation and Fundamental Theorems **167**

5.1 Expectation . . . 168
5.1.1 Properties of Expectation . . . 169
5.1.2 Expectation of Common Distributions . . . 171
5.1.3 Examples of Random Variables that Do Not Have Expectations . . . 174
5.1.4 Expectation of a Function . . . 175
5.1.5 Jensen's Inequality . . . 176
5.1.6 Expectation of Products of Random Variables . . . 180
5.1.7 Survivor Representation of Expectation . . . 181
5.1.8 Stopping Times . . . 181
5.1.9 Wald's Equality . . . 184
5.2 Expectations of Functions of Random Variables . . . 186
5.2.1 Variance . . . 189
5.2.2 Relationship Between Expectation and Variance . . . 190
5.2.3 Covariance . . . 191
5.3 Moment Generating Functions . . . 193
5.3.1 Probability Generating Functions . . . 195
5.3.2 Moment Generating Functions Determine Densities . . . 197
5.4 The Laplace Transform . . . 197
5.4.1 Common Transforms . . . 198
5.4.2 Properties of the Laplace Transform . . . 199
5.4.3 Solving Differential Equations with Laplace Transforms . . . 203
5.5 Fundamental Theorems . . . 204
5.5.1 Markov's Inequality . . . 205
5.5.2 Chebyshev's Inequality . . . 206
5.5.3 Chernoff's Bound . . . 207
5.5.4 Weak Law of Large Numbers . . . 210
5.5.5 Strong Law of Large Numbers** . . . 212
5.5.6 The Central Limit Theorem . . . 215
5.6 Summary of Chapter 5 . . . 219
5.6.1 Ending Historical Notes . . . 220
5.7 Problems for Chapter 5 . . . 225

II Stochastic Processes 233

6 The Poisson Process and Renewal Theory **235**

6.1 Binomial Processes . . . 237

6.1.1 Basic Properties of a Binomial Process . . . 238

6.1.2 Residual and Age Distributions for Binomial Processes . . . 239

6.1.3 Randomly Selected Times . . . 239

6.1.4 Derivation of the Joint Distribution . . . 240

6.1.5 Length Biasing . . . 242

6.2 Poisson Processes . . . 244

6.2.1 Basic Properties of a Poisson Process . . . 245

6.2.2 Relaxing Stationarity: Nonhomogenous Poisson Processes . . . 246

6.2.3 Relaxing Single Arrivals: Compound Poisson Processes . . . 247

6.2.4 Relaxing Independent Increments: Modulated Poisson Processes 248

6.2.5 Generalized Limit that Leads to the Poisson Process . . . 249

6.2.6 Residual and Age Distributions for Poisson Processes . . . 250

6.2.7 Superposition and Splitting of Poisson Processes . . . 252

6.3 Renewal Processes . . . 254

6.3.1 Limiting Number of Renewals per Unit Time . . . 257

6.3.2 Expected Number of Renewals . . . 257

6.3.3 Renewal Density . . . 260

6.3.4 Limiting Renewal Density . . . 261

6.3.5 Limiting Expected Number of Renewals per Unit Time . . . 262

6.3.6 Nonstationary and Nonindependent Increments . . . 263

6.3.7 Stationary and Independent Increments => Poisson Process . . . 264

6.4 Equilibrium Renewal Processes . . . 265

6.5 Joint Distribution of the Age and Residual Life . . . 267

6.5.1 Stationary Process for all Time . . . 268

6.5.2 Moments for the Age and Residual Life . . . 269

6.5.3 Length Biasing for General Renewal Processes . . . 269

6.6 Alternating Renewal Processes . . . 270

S

6.6.1 Time Averages . . . 273
6.7 Generalizations for Lattice Random Variables . . . 274
6.8 Summary of Chapter 6 . . . 275
6.9 Problems for Chapter 6 . . . 278

7 The M/G/1 Queue **283**
7.1 Elementary Systems Theory . . . 284
7.1.1 Time Averages and Customer Averages . . . 285
7.1.2 Almost Sure Limits . . . 285
7.1.3 Little's Law . . . 287
7.1.4 Level Crossing Law . . . 291
7.1.5 Poisson Arrivals See Time Averages . . . 293
7.2 Expected Values for the M/G/1 Queue . . . 295
7.2.1 Utilization of the M/G/1 Queue . . . 296
7.2.2 Expected Waiting and Response Times for the M/G/1 Queue . . . 296
7.2.3 General Observations on Queueing Systems . . . 300
7.2.4 Busy Period for M/G/1 Queues . . . 302
7.3 Distribution of Queue Length for the M/G/1 Queue . . . 304
7.3.1 General Observations on the Pollaczek-Khinchin Equation . . . 307
7.3.2 Distribution for Common M/G/1 Queues . . . 309
7.3.3 The M/M/1 Queue . . . 309
7.3.4 The $M/H_2(k)/1$ Queue . . . 310
7.3.5 The M/D/1 Queue . . . 312
7.3.6 The $M/E_k/1$ Queue . . . 313
7.3.7 Distribution of Response Time for M/G/1 Queues . . . 313
7.4 Distribution of the Number of Customers in a Busy Period . . . 316
7.4.1 Distribution of N_6 for the M/M/1 Queue . . . 317
7.5 Summary of Chapter 7 . . . 320
7.6 Problems for Chapter 7 . . . 324

8 Markov Processes 329
8.1 Markov Chains 333
8.1.1 The Chapman-Kolmogorov Equations 334
8.1.2 Classification of States 337
8.1.3 Mathematical Classification 339
8.1.4 The Stationary Distribution of Ergodic Chains 340
8.1.5 Balance Equations for Stationary Probabilities 343
8.1.6 Uniqueness of Stationary Probabilities 344
8.1.7 Guessing Stationary State Probabilities 348
8.1.8 A Note on Periodic Chains 349
8.2 Global Balance Equations 350
8.3 Semi-Markov Processes 352
8.3.1 Elimination of Self-Loops in Markov Chains 355
8.3.2 Semi-Markov Processes Are Not Markovian 356
8.4 Markov Processes 356
8.4.1 The Generator Matrix 356
8.4.2 The Stationary Distribution of a Markov Process 357
8.5 Variations of the M/M/l Queue 370
8.5.1 The M/M/l/k Queue 371
8.5.2 The M/M/k Queue 371
8.5.3 The M/M/k/k Queue 372
8.5.4 The M/M/l//p Queue 372
8.6 The G/M/1 Queue 373
8.7 Uniformized Markov Processes 376
8.8 The Chapman-Kolmogorov Differential Equations 378
8.9 Summary of Chapter 8 382
8.10 Problems for Chapter 8 384

9 Matrix Geometric Solutions **391**

9.1 Matrix Geometric Systems . . . 392

9.1.1 A Matrix Geometric Example . . . 393

9.1.2 Quasi Birth-Death Processes . . . 397

9.2 General Matrix Geometric Solutions . . . 399

9.3 Matrix Geometric Solutions for Markov Chains . . . 405

9.3.1 Uniformization for Matrix Geometric Markov Processes . . . 406

9.4 Properties of Solutions . . . 409

9.4.1 Stability . . . 409

9.4.2 Interpretation of the Rate Matrix . . . 411

9.4.3 Computational Properties . . . 417

9.4.4 Specifying Matrices . . . 420

9.5 Phase Distributions . . . 421

9.5.1 Continuous Phase Distributions . . . 421

9.5.2 The Distribution of a Continuous Phase Distribution . . . 422

9.5.3 The PH-Markov Process and PH-Renewal Process . . . 423

9.5.4 Discrete Phase Distributions . . . 424

9.6 Summary of Chapter 9 . . . 424

9.7 Problems for Chapter 9 . . . 426

10 Queueing Networks **429**

10.1 Reversibility . . . 431

10.1.1 Reversible Processes . . . 433

10.1.2 The Reversed Process . . . 434

10.1.3 Detailed Balance . . . 438

10.1.4 Stationary Distribution of a Reversible Process . . . 440

10.1.5 Kolmogorov's Criterion . . . 440

10.2 Properties of Reversible Processes . . . 441

10.2.1 Scaling of Transition Rates . . . 441

10.2.2 Spanning Tree Representation of a Reversible Process . . . 442

10.2.3 Joining of Two Independent Reversible Processes . . . 442

10.2.4 State Truncation in Reversible Processes . . . 443
10.3 Input-Output Properties . . . 445
10.3.1 Input-Output Properties of Reversible Queues . . . 445
10.3.2 Tandem Reversible Queues . . . 446
10.3.3 The Tandem Queue Is Not Reversible . . . 449
10.3.4 Poisson Arrivals and Departures ^ Product Form . . . 450
•10.3.5 Features of Product Form Networks . . . 455
10.3.6 Input-Output Properties of Quasi-Reversible Queues . . . 456
10.3.7 Quasi-Reversibility and Reversibility . . . 457
10.3.8 Multiple-Class Quasi-Reversible M/M/l Queue . . . 458
10.4 Symmetric Queues . . . 460
10.4.1 Quasi-Reversible Symmetric Queues . . . 462
10.4.2 Distribution of Number of Customers in a Symmetric Queue . . . 465
10.-5 Partial Balance and Quasi-Reversible Queues . . . 468
10.5.1 Local Balance . . . 469
10.5.2 General Definition of Partial Balance . . . 469
10.5.3 State Truncation Property . . . 470
10.5.4 Summary of Reversibility and Quasi-Reversibility . . . 471
10.6 Properties of Partial Balance and Product Form Networks . . . 472
10.6.1 Description of the Network . . . 472
10.6.2 Markov Routing Policy . . . 473
10.6.3 Types of Transitions . . . 474
10.6.4 The Distribution Property . . . 476
10.6.5 Open Product Form Networks . . . 477
10.6.6 Transition Rates for the Reversed Process . . . 477
10.6.7 Distribution of Number in Queue . . . 481
10.6.8 Distribution of Chains . . . 482
10.6.9 Closed Product Form Networks . . . 483
10.6.10 Calculating the Normalization Constant . . . 486
10.6.11 The Arrival-Departure Property . . . 488
10.6.12 Arrival-Departure Property for Closed Quasi-Reversible Queues 490

10.6.13 Mean Value Analysis . . . 490
10.6.14 The State Aggregation Property . . . 492
10.6.15 Flow Equivalent Servers . . . 494
10.7 Summary of Chapter 10 . . . 494
10.8 Problems for Chapter 10 . . . 498

11 **Epilogue and Special Topics** **503**
11.1 Stochastic Comparisons . . . 504
11.2 Large Deviation Theory . . . 506
11.3 Bounds on the Response Time for a G/G/l Queue . . . 510

A **Types of Randomness** **513**
A.I Randomness: Physical Systems . . . 514
A.1.1 Intrinsic Probability . . . 516
A.2/Randomness: Deterministic Systems . . . 518
A.2.1 The Baker's Transformation . . . 518
A.2.2 Dynamical Systems . . . 521
A.3 Deterministic Randomness** . . . 522
A.3.1 Isomorphism Between Systems . . . 523
A.3.2 Random Newtonian Systems . . . 524
A.4 Summary of Appendix A . . . 525
A.5 Problems for Appendix A . . . 527

B **Combinatorial Equalities and Inequalities** **529**
B.I Noninteger Combinatorial Expressions . . . 529
B.2 Binomial Formula . . . 530
B.3 Stirling's (de Moivre's) Formula . . . 530
B.4 Bounds on Factorial Expressions . . . 531
B.5 Noninteger Factorials** . . . 532

C **Tables of Laplace Transforms and Generating Functions** **535**
C.0.1' Laplace Transforms . . . 535
C.I Generating Functions ;-' . . . 539

D Limits and Order Relationships **545**

D.I Limits 545

D.2 Order/Relationships 547

E List of Common Summations **549**

Bibliography **553**

References - **559**

Index of Notation **567**

Index . **577**