Chaos and Fractals
Mew Frontiers of Science

With 686 illustrations, 40 in color
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>VII</td>
</tr>
<tr>
<td>Authors</td>
<td>XI</td>
</tr>
<tr>
<td>Foreword</td>
<td>1</td>
</tr>
<tr>
<td><strong>Introduction:</strong> Causality Principle, Deterministic Laws and Chaos</td>
<td>9</td>
</tr>
<tr>
<td>1 The Backbone of Fractals: Feedback and the Iterator</td>
<td>15</td>
</tr>
<tr>
<td>1.1 The Principle of Feedback</td>
<td>17</td>
</tr>
<tr>
<td>1.2 The Multiple Reduction Copy Machine</td>
<td>23</td>
</tr>
<tr>
<td>1.3 Basic Types of Feedback Processes</td>
<td>27</td>
</tr>
<tr>
<td>1.4 The Parable of the Parabola — Or: Don’t Trust Your Computer</td>
<td>37</td>
</tr>
<tr>
<td>1.5 Chaos Wipes Out Every Computer</td>
<td>49</td>
</tr>
<tr>
<td>1.6 Program of the Chapter: Graphical Iteration</td>
<td>60</td>
</tr>
<tr>
<td>2 Classical Fractals and Self-Similarity</td>
<td>63</td>
</tr>
<tr>
<td>2.1 The Cantor Set</td>
<td>67</td>
</tr>
<tr>
<td>2.2 The Sierpinski Gasket and Carpet</td>
<td>78</td>
</tr>
<tr>
<td>2.3 The Pascal Triangle</td>
<td>82</td>
</tr>
<tr>
<td>2.4 The Koch Curve</td>
<td>89</td>
</tr>
<tr>
<td>2.5 Space-Filling Curves</td>
<td>94</td>
</tr>
<tr>
<td>2.6 Fractals and the Problem of Dimension</td>
<td>106</td>
</tr>
<tr>
<td>2.7 The Universality of the Sierpinski Carpet</td>
<td>112</td>
</tr>
<tr>
<td>2.8 Julia Sets</td>
<td>122</td>
</tr>
<tr>
<td>2.9 Pythagorean Trees</td>
<td>126</td>
</tr>
<tr>
<td>2.10 Program of the Chapter: Sierpinski Gasket by Binary Addresses</td>
<td>132</td>
</tr>
<tr>
<td>3 Limits and Self-Similarity</td>
<td>135</td>
</tr>
<tr>
<td>3.1 Similarity and Scaling</td>
<td>138</td>
</tr>
<tr>
<td>3.2 Geometric Series and the Koch Curve</td>
<td>147</td>
</tr>
<tr>
<td>3.3 Corner the New from Several Sides: Pi and the Square Root of Two</td>
<td>153</td>
</tr>
<tr>
<td>3.4 Fractals as Solution of Equations</td>
<td>168</td>
</tr>
<tr>
<td>3.5 Program of the Chapter: The Koch Curve</td>
<td>179</td>
</tr>
</tbody>
</table>
Table of Contents

4 Length, Area and Dimension: Measuring Complexity and Scaling Properties 183
  4.1 Finite and Infinite Length of Spirals 185
  4.2 Measuring Fractal Curves and Power Laws 192
  4.3 Fractal Dimension 202
  4.4 The Box-Counting Dimension 212
  4.5 Borderline Fractals: Devil's Staircase and Peano Curve 220
  4.6 Program of the Chapter: The Cantor Set and Devil's Staircase 226

Encoding Images by Simple Transformations 229
  5.1 The Multiple Reduction Copy Machine Metaphor 231
  5.2 Composing Simple Transformations 234
  5.3 Relatives of the Sierpinski Gasket 244
  5.4 Classical Fractals by IFSs 252
  5.5 Image Encoding by IFSs 258
  5.6 Foundation of IFS: The Contraction Mapping Principle 263
  5.7 Choosing the Right Metric 274
  5.8 Composing Self-Similar Images 278
  5.9 Breaking Self-Similarity and Self-Affinity or, Networking with MRCMs 283
  5.10 Program of the Chapter: Iterating the MRCM 293

6 The Chaos Game: How Randomness Creates Deterministic Shapes 297
  6.1 The Fortune Wheel Reduction Copy Machine 300
  6.2 Addresses: Analysis of the Chaos Game 307
  6.3 Tuning the Fortune Wheel 321
  6.4 Random Number Generator Pitfall 333
  6.5 Adaptive Cut Methods 341
  6.6 Program of the Chapter: Chaos Game for the Fern 350

7 Recursive Structures: Growing of Fractals and Plants 353
  7.1 L-Systems: A Language for Modeling Growth 357
  7.2 Growing Classical Fractals with MRCMs 364
  7.3 Turtle Graphics: Graphical Interpretation of L-Systems 376
  7.4 Growing Classical Fractals with L-Systems 380
  7.5 Growing Fractals with Networked MRCMs 392
  7.6 L-System Trees and Bushes 397
  7.7 Program of the Chapter: L-systems 402

8 Pascal's Triangle: Cellular Automata and Attractors 407
  8.1 Cellular Automata 412
  8.2 Binomial Coefficients and Divisibility 423
  8.3 IFS: From Local Divisibility to Global Geometry 434
  8.4 HIFS and Divisibility by Prime Powers 442
  8.5 Catalytic Converters or how many Cells are Black 450
  8.6 Program of the Chapter: Cellular Automata 454
**Table of Contents**

9 Irregular Shapes: Randomness in Fractal Constructions 457
   9.1 Randomizing Deterministic Fractals, 459
   9.2 Percolation: Fractals and Fires in Random Forests, 463
   9.3 Random Fractals in a Laboratory Experiment, 475
   9.4 Simulation of Brownian Motion, 481
   9.5 Scaling Laws and Fractional Brownian Motion, 491
   9.6 Fractal Landscapes, 497
   9.7 Program of the Chapter: Random Midpoint Displacement, 503

10 Deterministic Chaos: Sensitivity, Mixing, and Periodic Points 507
   10.1 The Signs of Chaos: Sensitivity, 509
   10.2 The Signs of Chaos: Mixing and Periodic Points, 520
   10.3 Ergodic Orbits and Histograms, 525
   10.4 Paradigm of Chaos: The Kneading of Dough, 536
   10.5 Analysis of Chaos: Sensitivity, Mixing, and Periodic Points, 549
   10.6 Chaos for the Quadratic Iterator, 560
   10.7 Mixing and Dense Periodic Points Imply Sensitivity, 569
   10.8 Numerics of Chaos: Worth the Trouble or Not?, 575
   10.9 Program of the Chapter: Time Series and Error Development, 581

11 Order and Chaos: Period-Doubling and its Chaotic Mirror 585
   11.1 The First Step From Order to Chaos: Stable Fixed Points, 592
   11.2 The Next Step From Order to Chaos: The Period Doubling Scenario, 603
   11.3 The Feigenbaum Point: Entrance to Chaos, 619
   11.4 From Chaos to Order: a Mirror Image, 628
   11.5 Intermittency and Crises: The Backdoors to Chaos, 640
   11.6 Program of the Chapter: Final State Diagram, 651

12 Strange Attractors: The Locus of Chaos 655
   12.1 A Discrete Dynamical System in Two Dimensions: Henon’s Attractor, 659
   12.2 Continuous Dynamical Systems: Differential Equations, 678
   12.3 The Rossler Attractor, 686
   12.4 The Lorenz Attractor, 697
   12.5 Quantitative Characterization of Strange Chaotic Attractors: Ljapunov Exponents, 709
   12.6 Quantitative Characterization of Strange Chaotic Attractors: Dimensions, 721
   12.7 The Reconstruction of Strange Attractors, 745
   12.8 Fractal Basin Boundaries, 757
   12.9 Program of the Chapter: Rossler Attractor, 766

13 Julia Sets: Fractal Basin Boundaries 769
   13.1 Julia Sets as Basin Boundaries, 771
   13.2 Complex Numbers — A Short Introduction, 776
   13.3 Complex Square Roots and Quadratic Equations, 784
   13.4 Prisoners versus Escapees, 789
   13.5 Equipotentials and Field Lines for Julia Sets, 800
   13.6 Binary Decomposition, Field Lines and Dynamics, 812
# Table of Contents

13.7 Chaos Game and Self-Similarity for Julia Sets, .............................. 820
13.8 The Critical Point and Julia Sets as Cantor Sets, ........................ 826
13.9 Quaternion Julia Sets, ......................................................... 837
13.10 Program of the Chapter: Julia Sets ....................................... 839

14 The Mandelbrot Set: Ordering the Julia Sets .................................. 841
14.1 From the Structural Dichotomy to the Binary Decomposition .......... 843
14.2 The Mandelbrot Set — A Road Map for Julia Sets ........................ 855
14.3 The Mandelbrot Set as a Table of Content ................................ 878
14.4 Program of the Chapter: The Mandelbrot Set ............................ 896

A A Discussion of Fractal Image Compression .................................... 903
   Yuval Fisher
   A.1 Self-Similarity in Images .................................................. 906
   A.2 A Special MRCM ............................................................. 908
   A.3 Encoding Images ............................................................ 912
   A.4 Ways to Partition Images .................................................. 914
   A.5 Implementation Notes ....................................................... 917

B Multifractal Measures ............................................................... 921
   Carl J. G. Evertsz and Benoit B. Mandelbrot
   B.1 Introduction ................................................................. 922
   B.2 The Binomial and Multinomial Measures ................................ 927
   B.3 Methods for Estimating the Function \( f(a) \) from Data .......... 938
   B.4 Probabilistic Roots of Multifractals. Role of \( f(a) \) in Large Deviation-Theory .............................. 944
   B.5 Some Applications, and Advanced Multifractals ..................... 952

Bibliography ................................................................................. 955

Index ......................................................................................... 971